On Correspondences between Probabilistic First-Order and Description Logics

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Abstract This paper analyzes the probabilistic description logic P-$\mathcal{SHIQ}$ by looking at it as a fragment of probabilistic first-order logic with semantics based on possible worlds. We argue that this is an appropriate way of investigating its properties and developing extensions. We show how the previously made arguments about different types of first-order probabilistic semantics apply to P-$\mathcal{SHIQ}$. This approach has advantages for the future of both P-$\mathcal{SHIQ}$, which can further evolve by incorporating semantic theories developed for the full first-order case, and the first-order logic, for which very few interesting decidable fragments are currently known. The paper also presents a probabilistic logic P-$\mathcal{SHIQ}^+$ which addresses some of the identified limitations of P-$\mathcal{SHIQ}$.

1 Introduction and Preliminaries

A common complaint about Description Logics (DLs), especially as the basis for ontology languages, is that they fail to support non-classical representations of uncertainty. While DLs typically can represent and reason with some forms of uncertainty (e.g., with existential or disjunctive information) in a variety of ways , until recently they have not handled probability.

One answer to this complaint is the P-$\mathcal{SH}$ family of logics which allow for the incorporation of probabilistic formulae as an extension of the familiar and widely used $\mathcal{SH}$ DLs. Unlike Bayesian extensions to DLs (e.g., [1]), the P-$\mathcal{SH}$ family consists of purely logical extensions to the semantics and inference services. These logics are also decidable, generally of the same worst case complexity as the base logic, and can be implemented on top of existing DL reasoners [2].

However, there are several issues with the P-$\mathcal{SH}$ family both from an expressivity and from a theoretical point of view. First, it has not been fully clear whether it properly combines statistical and subjective probabilities. Second, probabilistic ABoxes have a number of strong and strange restrictions (including no probabilistic roles and only one probabilistic individual per ABox). Finally, the semantics of the P-$\mathcal{SH}$ family (in terms of possible worlds) is not particularly familiar in the DL setting.

It is standard to analyze DLs by considering them as fragments of first-order logic. In this paper, we apply this methodology to the analysis of the P-$\mathcal{SH}$ family of probabilistic DLs by considering them as fragments of FOPL_2

(FOL with possible world based semantics [3]). This analysis yields insight into the semantics of P-$\mathcal{SHIQ}$ and into principled ways of extending it to address troublesome limitations.

1.1 P-$\mathcal{SHIQ}$

P-$\mathcal{SHIQ}$ [4] is a probabilistic generalization of the DL $\mathcal{SHIQ}$. It has a number of properties that make it an attractive formalism for managing uncertainty in OWL ontologies. First, it is very expressive as it supports arbitrary $\mathcal{SHIQ}$ concepts in probabilistic axioms. Second, any $\mathcal{SHIQ}$ ontology can be used as a basis for a P-$\mathcal{SHIQ}$ ontology which facilitates transition from classical to probabilistic ontological models [5]. Thirdly, its reasoning tasks are decidable.

Syntax of P-$\mathcal{SHIQ}$ The syntax of P-$\mathcal{SHIQ}$ is an extension of the syntax of $\mathcal{SHIQ}$ with *conditional constraints* [6]. Conditional constraints are expressions of the form $(D|C)[l, u]$ where C and D are $\mathcal{SHIQ}$ concepts[1]. Conditional constraints can be used for representing uncertainty in both terminological (TBox) and assertional (ABox) knowledge. A probabilistic TBox (PTBox) is a 2-tuple $PT = (\mathcal{T}, \mathcal{P})$ where $\mathcal{T}$ is a $\mathcal{SHIQ}$ TBox and $\mathcal{P}$ is a finite set of default[2] conditional constraints. Informally, a PTBox axiom $(D|C)[l, u]$ means that "generally, if a randomly chosen individual belongs to C, its probability of belonging to D is between l and u". A probabilistic ABox (PABox) is a finite set of strict conditional constraints pertaining to a single probabilistic individual o. All constraints in a PABox are of the restricted form $(D|\top)[l, u]$. Informally, they mean that "o is a member of D with probability between l and u" [4].

Semantics and Reasoning with P-$\mathcal{SHIQ}$ Although the semantics of P-$\mathcal{SHIQ}$ is standardly given in terms of *possible worlds* [4], we present a slightly different formulation based on a closely related notion of *types* [7]. This will help to ensure uniform use of terminology in the later sections. Two preliminary notions are needed:

Definition 1 (Closure, Concept Type). *For an ontology $\mathcal{O} = (\mathcal{T}, \mathcal{A})$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox, the concept closure of $\mathcal{T}$, denoted as $cl(\mathcal{T})$, is the smallest set of concepts such that:*

- *$\top$ is in $cl(\mathcal{T})$,*
- *For all $C \sqsubseteq D \in \mathcal{T}$, C and D are in $cl(\mathcal{T})$,*
- *If $C \in cl(\mathcal{T})$ then its subconcepts are in $cl(\mathcal{T})$,*
- *If $C \in cl(\mathcal{T})$ then $\dot\neg C$ (negation in the negation normal form) is in $cl(\mathcal{T})$.*

A concept type Ct of an ontology $\mathcal{O}$ is a subset of $cl(\mathcal{T})$ such that for all $C \in Ct$, $C \in Ct$ iff $\dot\neg C \notin Ct$. Given an interpretation $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}) \models \mathcal{T}$, a domain element $e \in \Delta^{I}$ is an instance of type Ct iff $e \in C^{\mathcal{I}}$ for all $C \in Ct$.

[1] Without loss of generality we will further assume that C, D are atomic concepts.

[2] Default aspects of P-$\mathcal{SHIQ}$ are immaterial in the context of this paper.

The relation $ctype(e) = \{Ct|e$ is instance of $Ct\}$ is a well-defined function. The set $\{Ct \subseteq cl(\mathcal{T}) | \bigsqcap_{C_i \in Ct} C_i$ is satisfiable w.r.t. $\mathcal{T}\}$ is denoted $Types(\mathcal{O})$.

Definition 2 (Type Consistency). *A type Ct is said to be consistent with a concept C ($Ct \dashv\vdash C$) if $C \in Ct$.*

It is convenient to define probabilistic models using types. A probabilistic interpretation Pr is a function $Pr : Types(\mathcal{O} \rightarrow [0,1])$ such that $\sum_{T \in Types(\mathcal{O})} Pr(T) = 1$. The probability of a concept C, denoted as $Pr(C)$, is defined as $\sum_{T \dashv\vdash C} Pr(T)$. $Pr(D|C)$ is used as an abbreviation for $Pr(C \sqcap D)/Pr(C)$ given $Pr(C) > 0$. A probabilistic interpretation Pr satisfies a conditional constraint $(D|C)[l,u]$ (or Pr is a model of $(D|C)[l,u]$) denoted as $Pr \models (D|C)[l,u]$ iff $Pr(C) = 0$ or $Pr(D|C) \in [l,u]$. Finally, Pr satisfies a set of conditional constraints $\mathcal{P}$ iff it satisfies each of the constraints.

The following are the core reasoning problems of P-$\mathcal{SHIQ}$ [8]:

- *Probabilistic Satisfiability (PSAT).* PSAT is the problem of deciding whether there exists a probabilistic interpretation that satisfies given PTBox.
- *Tight Logical Entailment (TLogEnt).* TLogEnt is the problem of computing the tightest probability intervals for logical consequences.

1.2 First-Order Probabilistic Logic

FOPL_2 is a probabilistic generalization of first-order logic aimed at capturing belief statements (the subscript 2 stands for the Type 2 semantics [3]), like "the probability that Tweety (a particular bird) flies is over 90%" [3]. It is very expressive allowing to attach probabilities to arbitrary first-order formulas (both closed and open). Its representational and computational properties have been well investigated, and the results are applicable to its fragments.

Syntax of FOPL_2 The syntax of FOPL_2 is defined as follows [3]: assume a first-order alphabet Φ of function and predicate names, and a countable set of object variables X^o. *Object terms* are formed by closing X^o off under function application as usual. The language also contains *field terms*, which range over reals (with 0 and 1 being distinguished constants), and probability terms of the form $w(\phi)$, where ϕ is a first-order formula. Field terms are closed off under applications of functions $\times, +$ on reals (e.g., $t_1 + t_2$ is a field term iff t_1, t_2 are). Then FOPL_2 formulas are defined as follows:

- If P is an n-ary predicate name in Φ and $t_1, \ldots, t_n$ are object terms, then $P(t_1, \ldots, t_n)$ is an atomic formula.
- If t_1, t_2 are field terms, then $t_1 \leq t_2$, $t_1 \geq t_2$, $t_1 < t_2$, $t_1 > t_2$, $t_1 = t_2$ are atomic formulas. Standard relationships between (in)equality relations are assumed to hold.
- If ϕ, ψ are formulas and $x \in X^o$, then $\phi \wedge \psi$, $\phi \vee \psi$, $\forall(x)\phi$, $\exists(x)\phi$, $\neg\phi$ are formulas. Standard relationships between logical connectives and quantifiers are assumed to hold.

Finally, the denotation $w(\phi|\psi) \leq t$ is the abbreviation of $w(\phi \wedge \psi) \leq t \times w(\psi)$.

Semantics of $\mathbf{FOPL_2}$ A probabilistic interpretation (Type 2 probability structure in [3]) M is a tuple (D, S, π, μ), where D is a domain, S is a set of states (sometimes called *possible worlds*), π is a function $S \times \Phi \rightarrow \Phi_D$ (where Φ_D is a set of predicates and functions over D) which preserves arity, and μ is a probability distribution over S. Let v be valuation function that maps each object variable to an element of D. Then M together with a state s and a valuation v associates each object term t with an element of D ($[t]_{M,s,v} \in D$) and each field term t with a real number. (M, s, v) also associate formulas with truth values as follows (we write $(M, s, v) \models \phi$ iff ϕ is true in M):

Field terms of the form $w(\phi)$ are associated as follows: $[w(\phi)]_{M,v} = \mu\{s \in S | (M, s, v) \models \phi\}$.

- $(M, s, v) \models P(x)$ if $v(x) \in \pi(s, P)$.
- $(M, s, v) \models t_1 < t_2$ if $[t_1]_{M,s,v} < [t_2]_{M,s,v}$.
- $(M, s, v) \models \forall(x)\phi$ if $(M, s, v[x/d]) \models \phi$ for all $d \in D$.

Other formulas, e.g. $\phi \wedge \psi$, $\neg\psi$, $t_1 = t_2$, etc. are defined as usual. It remains to define the mapping for the field terms of the form $w(\phi)$: $[w(\phi)]_{M,v} = \mu\{s' \in S | (M, s', v) \models \phi\}$ (note that the association does not depend on a state here).

As usual, a $\mathrm{FOPL_2}$ formula is called *satisfiable* if there exists an interpretation and a valuation in which the formula is true. It is called *valid* (denoted $\models \phi$) iff it is satisfied by all interpretations and valuations. More detailed exposition of $\mathrm{FOPL_2}$ 2 can be found in [3].

2 Correspondences between P-$\mathcal{SHIQ}$ and $\mathbf{FOPL_2}$

This section presents a translation between P-$\mathcal{ALCI}$ and the $\mathrm{FOPL_2}$. For the sake of brevity we will limit our attention to $\mathcal{ALCI}$ concepts as the translation can be extended to $\mathcal{SHIQ}$ concepts in a straightforward way (inverse roles are included because they will be important in the following section). We will show that the translation preserves satisfiability of formulas so that P-$\mathcal{SHIQ}$ can be viewed as a fragment of $\mathrm{FOPL_2}$.

2.1 Syntactic Translation

We first define the injective function κ to be the mapping of P-$\mathcal{ALCI}$ formulas to $\mathrm{FOPL_2}$. It is a superset of the standard translation of $\mathcal{ALCI}$ formulas into the formulas of FOL [9] (in the Table 2.1 A, B stand for concept names, R for role names, C, D for concept expressions, P for role names or inverses, $var \in \{x, y\}; var' = x$ if $var = y$ and y if $var = x$).

This function transforms a P-$\mathcal{ALCI}$ ontology into the collection of $\mathrm{FOPL_2}$ theories. More precisely, a PTBox is transformed into a set of formulas while each PABox is translated into its own $\mathrm{FOPL_2}$ theory. This is because PABox axioms in P-$\mathcal{SHIQ}$ are treated as PTBox axioms (just labeled with the individual name) so they do *not* correspond to ground formulas in $\mathrm{FOPL_2}$. Interestingly, in our extension of P-$\mathcal{SHIQ}$ (see Section 3) they do (as they should).

Table 1. Translation of P-$\mathcal{ALCI}$ formulae into FOPL_2

P-$\mathcal{ALCI}$	FOPL_2
$\kappa(A, var)$	$A(var)$
$\kappa(\neg C, var)$	$\neg(\kappa(C, var))$
$\kappa(R, var, var')$	$R(var, var')$
$\kappa(R^-, var, var')$	$R(var', var)$
$\kappa(C \sqcap D, var)$	$\kappa(C, var) \wedge \kappa(D, var)$
$\kappa(C \sqcup D, var)$	$\kappa(C, var) \vee \kappa(D, var)$
$\kappa(\forall P.C, var)$	$\forall(var')(\kappa(P, var, var') \rightarrow \kappa(C, var'))$
$\kappa(\exists P.C, var)$	$\exists(var')(\kappa(P, var, var') \wedge \kappa(C, var'))$
$\kappa(a : C)$	$\kappa(C, x)[a/x]$
$\kappa((a, b) : P)$	$\kappa(P, x, y)[a/x, b/y]$
$\kappa(C \sqsubseteq D, x)$	$\forall(x)(\kappa(C, x) \rightarrow \kappa(D, x))$
$\kappa((B\|A)[l, u], x)$	$l \leq w(B(x)\|A(x)) \leq u$

2.2 Satisfiability Preservation

Theorem 1. *A P-$\mathcal{SHIQ}$ PTBox $PT = (\mathcal{T}, \mathcal{P})$ is satisfiable iff the corresponding $FOPL_2$ theory $\{\kappa(\phi)|\phi \in \mathcal{T} \cup \mathcal{P}\}$ is satisfiable.*

Proof. The proof will proceed by construction, i.e. we will show that a satisfying probabilistic model in P-$\mathcal{SHIQ}$ can be transformed into a satisfying probabilistic interpretation in FOPL_2 and vice versa.

($\Rightarrow$): Assume there exists $Pr : Types(\mathcal{O}) \rightarrow [0, 1]$ s.t. $Pr \models \mathcal{T}$ and $Pr \models \mathcal{P}$. We will construct $M = (D, S, \pi, \mu)$ s.t. $M \models \kappa(\phi)$ for all $\phi \in \mathcal{T} \cup \mathcal{P}$. For that we first observe that the existence of Pr implies the existence of a $\mathcal{SHIQ}$ interpretation: $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}) \models \mathcal{T}$ [4]. Thus we can define D and π as follows: $D = \Delta^{\mathcal{I}}$; $\pi(t) = (\kappa^{-1}(t))^{\mathcal{I}}$ for all unary and binary predicates t. Note that π interprets all predicate names independently of a state.

The key is to construct the set S and ensure the following property:

$$Pr(\{Ct \in Types(\mathcal{O})|Ct \dashv\vdash C\}) = \mu(\{s \in S|(M, s, v) \models \kappa(C)\}) \qquad (1)$$

for any concept C and some valuation v

Property (1) guarantees that $Pr(C) = \mu(\kappa(C))$ for all $(C \in cl(\mathcal{T})$ (hence $Pr \models (D|C)[l, u]$ implies $(M, v) \models l \leq w(\kappa(D, x)|\kappa(C, x)) \leq u$ for some v). It can be achieved by defining a straightforward bijection between types and states using κ. For each type $Ct = \{C_1, \ldots, C_k\}$ we define a state $s = \{\kappa(C_1, x), \ldots, \kappa(C_k, x)\}$ (we use $s = \kappa(Ct)$ as a shorthand notation). It is not hard to see that, if $\bigsqcap_i C_i$ is satisfiable w.r.t. $\mathcal{T}$ then $\bigwedge_i \kappa(C_i, x)$ is satisfiable w.r.t. $\kappa(\mathcal{T})$.

Similarly to P-$\mathcal{SHIQ}$ we use the notation $s \dashv\vdash \phi$ where ϕ is a translation of some $\mathcal{ALCI}$ concept expression into FOPL_2 iff $\phi \in s$. We leave out the demonstration that $Ct \dashv\vdash C$ implies $\kappa(Ct) \dashv\vdash \kappa(C, x)$.

Finally, we define the probability function: $\mu(s) = Pr(\kappa^{-1}(s))$ for all $s \in S$.

It remains to show that $M = (D, S, \pi, \mu)$ satisfies the corresponding formula in $FOPL_2$ and some valuation v. For the sake of brevity we will only show this for probabilistic axioms. Consider a PTBox axiom $(B|A)[l, u]$. It is translated into the formula $l \leq w(\kappa(B, x)|\kappa(A, x)) \leq u$. Now $\frac{\sum_{Ct \dashv\vdash B \sqcap A} P(Ct)}{\sum_{Ct \dashv\vdash A} P(Ct)} \in [l, u]$, therefore $\frac{\sum_{s \dashv\vdash \kappa(B,x) \wedge \kappa(A,x)} \mu(s)}{\sum_{s \dashv\vdash \kappa(A,x)} \mu(s)} \in [l, u]$ (by definition of S and μ). In order to show that this is equivalent to $\frac{\mu\{s \in S:(M,s,v) \models B(x) \wedge A(x))\}}{\mu\{s \in S:(M,s,v) \models A(x)\}} \in [l, u]$ it suffices to prove that, if there exists a state $s' \in S$ in M s.t. $s' \dashv\vdash c(x)$ (c is a unary predicate) then there exists a valuation v s.t. $(M, s', v) \models c(x)$. Take some $s' \in S$. Then there exists $v : (M, s', v) \models c_1(a) \wedge \cdots \wedge c_k(a) \wedge \kappa(\mathcal{T})$ where a is a fresh constant (this follows from consistency of concept types). Since $c = c_i \in s'$ for some i it follows that $(M, s', v[a/x]) \models c(x)$.

($\Leftarrow$) can be shown completely analogously, so we skip the details.

□

2.3 Why the Translation Matters?

This translation provides a new insight into both P-$\mathcal{SHIQ}$ and $FOPL_2$ being valuable for the following reasons:

Firstly, while it has been claimed that P-$\mathcal{SHIQ}$ can represent and reason about both statistical and subjective (i.e. degrees of belief) knowledge [4], it is vulnerable to all arguments regarding the issues with the possible world semantics of $FOPL_2$ and its inappropriateness for handling statistics [10, 11, 3]. The most serious issue is that generic probabilistic statements that are aimed at representing statistics do *not* fulfill this role. The proper statistical interpretation of a statement $(D|C)[l, u]$ would be: "if a randomly chosen *domain element* belongs to C, its probability of belonging to D is in $[l, u]$. However, in P-$\mathcal{SHIQ}$ semantics it instead quantifies over randomly chosen *named individuals.* The difference is illustrated on the following example:

Example 1 (Inadequate Handling of Statistics). Consider the PTBox: $\mathcal{T} = \{A \equiv (= 1R.B), B \equiv (= 1R^{-}.A)\}; \mathcal{P} = \{(A|\top)[0.5, 0, 5]\}$. The TBox component implies a bijection between the interpretations of A and B so it seems that if the probability that a random object is in A is 0.5 then the probability of being in B should also be 0.5 (because there are as many instances of B in the domain as there are of A). However, the result is $[0, 1]$ because this relationship between the interpretations is *not* captured in the possible world based semantics.

Secondly, the translation helps to understand the limitations of P-$\mathcal{SHIQ}$, namely, the inability to combine probabilistic knowledge about multiple individuals in a single ABox, unnatural separation of classical and probabilistic individuals, inability to represent probabilistic relationships between roles, etc. The common reason behind all those issues is that the set of states S is quite coarse-grained and contains only information about concepts, but not roles or named individuals (constants). We will take it up in Section 3 in which extensions to P-$\mathcal{SHIQ}$ are proposed.

Finally, the translation is also valuable for the analysis of FOPL_2. Since it is a highly undecidable logic it is natural to analyze its interesting decidable fragments and P-$\mathcal{SHIQ}$ is one of such fragments. It is known that all fragments of FOPL_2 that have bounded model property are decidable. Interestingly P-$\mathcal{SHIQ}$ does not have such property (because $\mathcal{SHIQ}$ does not have finite model property) but it *is* decidable which suggests that there might be other criteria for decidability. In particular, it is known that given the unrestricted use of variables it is undecidable even with a single unary predicate.

3 Possible extensions to P-$\mathcal{SHIQ}$

P-$\mathcal{SHIQ}$ has a number of limitations which are caused by two main reasons. First, it does not make use of all features that can be supported by the FOPL_2 semantics. Second, as briefly explained in the previous section, possible world semantics is not the best ground for representing and reasoning with statistical knowledge. In this paper we attempt to address the first type of limitations (the resulting logic will be called P-$\mathcal{SHIQ}^+$). The important issues, which will be addressed in P-$\mathcal{SHIQ}^+$, are the following:

Separation between classical and probabilistic individuals. In P-$\mathcal{SHIQ}$ all named individuals are split onto those for which there is some probabilistic knowledge (i.e. PABox axioms) and those for which there is not. As a result, there is no straightforward way to combine classical and probabilistic reasoning for a single individual although that seems a very natural thing to do. For example, consider the following knowledge base:

Example 2. $\{Big \sqcap Dense \sqsubseteq Heavy, ball : Big, (ball : Dense)[0.9, 1]\}$

Intuitively the answer to the query $ball : (Heavy)[?, ?]$ should be $[0.9, 1]$. However, due to the separation, the two *ball*s are regarded as distinct individuals so such entailment is not supported. There are, of course, workarounds, for example, it is possible to do some sort of “punning” between the two balls and obtain the desired result.

Isolated PABoxes. P-$\mathcal{SHIQ}$ does not allow for the representation of probabilistic role assertions between *probabilistic* individuals. In other words, while it is possible to state that $(jim : Parent)[0.9, 0.95]$ it is not possible to state that $((jim, pete) : parentOf)[0.9, 0.95]$. There is a way to represent probabilistic role assertions in P-$\mathcal{SHOIN}$ but only between a classical and a probabilistic individual [4]. Thus it is not possible to assert a probabilistic role between individuals for whom we store other probabilistic knowledge. Consider the example:

Example 3. $\{john : \neg Tall[0.1, 0.2], pete : Tall[0.9, 1],$
$(john, pete) : friendOf[0.8, 1]\}$

In P-$\mathcal{SHIQ}$ either *john* or *pete* will have to be a classical individual so we will not be able to use all the available probabilistic knowledge to answer the query

$john : (Tall \sqcap \exists friendOf.\neg Tall)[?, ?]$ (the probability that John is tall but has a short friend).

No support of probabilistic role hierarchies. P-$\mathcal{SHIQ}$ only supports generic probabilistic relationships between concepts but not roles.

We next proceed to the syntax and semantics of P-$\mathcal{SHIQ}^+$.

3.1 Syntax of P-$\mathcal{SHIQ}^+$

Syntactically, P-$\mathcal{SHIQ}^+$ differs from P-$\mathcal{SHIQ}$ in two main aspects. Firstly, the notion of a conditional constraint is extended. Secondly, instead of a collection of independent PABoxes in P-$\mathcal{SHIQ}$, a P-$\mathcal{SHIQ}^+$ ontology contains only a single probabilistic component, namely, a collection of conditional constraints. One preliminary definition is needed.

Definition 3 (Entity). *An entity in P-$\mathcal{SHIQ}^+$ is one of: role, concept, concept assertion, role assertion.*

Intuitively, entities are the things that can appear in conditional constraints. Now we can define the extended syntax of conditional constraints.

Definition 4 (Conditional Constraint). *A conditional constraint in P-$\mathcal{SHIQ}^+$ is an expression of the form: $(\psi|\phi)[l, u]$ where ψ, ϕ are entities.*

Note that every conditional constraint in P-$\mathcal{SHIQ}$ is a well-formed conditional constraint in P-$\mathcal{SHIQ}^+$. No other probabilistic formulas are allowed. A probabilistic ontology in P-$\mathcal{SHIQ}^+$ has a simpler structure than in P-$\mathcal{SHIQ}$:

Definition 5 (Probabilistic Ontology). *A probabilistic ontology in P-$\mathcal{SHIQ}^+$ is a tuple $PO = (\mathcal{O}, \mathcal{P})$ where $\mathcal{O} = (\mathcal{T}, \mathcal{A})$ is a $\mathcal{SHIQ}$ ontology*[3] *and $\mathcal{P}$ is a finite collection of conditional constraints.*

Any P-$\mathcal{SHIQ}$ ontology can be represented as a P-$\mathcal{SHIQ}^+$ ontology, however, the representation of PABox constraints is a bit different. Instead of implicitly attributing statements of the form $(C|\top)[l, u]$ to a particular individual, it is represented as $(C(a)|\top)[l, u]$ and becomes an element of $\mathcal{P}$. In other words, PABox constraints in P-$\mathcal{SHIQ}^+$ correspond to ground probabilistic formulas in $FOPL_2$ while in P-$\mathcal{SHIQ}$ they do not.

3.2 Semantics of P-$\mathcal{SHIQ}^+$

From an analysis of P-$\mathcal{SHIQ}$ alone, it may not be immediately clear that its drawbacks can be addressed. However, looking at the corresponding fragment of $FOPL_2$ immediately reveals that none of the limitations is fundamental. $FOPL_2$ does not require any separation between constants and neither does it prohibit attaching probabilities to binary predicates. In fact, what is necessary from the $FOPL_2$ point of view is that the specification of the states provides enough

[3] Role hierarchies are assumed to be a part of the TBox.

information to determine if a particular formula is true at the given state (to be able to assign probabilities to weighted formulas). P-$\mathcal{SHIQ}$ specification only allows that for unary predicates and their combinations. We are going to extend it to binary predicates as well. It is worth noting that FOPL_2 does not impose restrictions on the set of states, so one is free to extend it in an arbitrary way.

Recall from Section 1 that states in P-$\mathcal{SHIQ}$ are types, that is, subsets of the closure of relevant concepts. Unfortunately types do not provide enough information about relationships between domain objects. Our idea is to extend single type structures to two type structures, which we call *links*. Informally, a link is a specification of an ordered pair of domain elements which includes types of both as well as relationships between them.

Definition 6 (Extended Concept Type). *Concept types in P-$\mathcal{SHIQ}^+$ are extensions of P-$\mathcal{SHIQ}$ concept types which may contain individual names. They have to satisfy one additional requirement, namely, if $a \in Ct$ and $a : C \in \mathcal{O}$ then $C \in Ct$.*

ctype is also extended: $ctype(e) = Ct$ if additionally $e = a^{\mathcal{I}}$ for all $a \in Ct$.

Definition 7 (Role type). *Let $\mathcal{R}$ be the set of roles in an ontology $\mathcal{O} = (\mathcal{T}, \mathcal{A})$ (i.e. role names and their inverses) that can participate in probabilistic statements. Then a role type Rt is a subset of $\mathcal{R}$ that satisfies the following condition: if $R \in Rt$ and $R \sqsubseteq S \in \mathcal{O}$ or $Inv(R) \sqsubseteq Inv(S) \in \mathcal{O}$ then $S \in Rt$.*

Given an interpretation $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}) \models \mathcal{O}$, a pair of domain elements (e, f) is an instance of a role type Rt if $(e, f) \in R^{\mathcal{I}}$ for all $R \in Rt$.

Analogously to *ctype*, the function $rtype(e, f) = \{Rt | (e, f) \text{ is instance of } Rt\}$. Observe that it is also a well-defined, total function.

Definition 8 (Link). *A link L is a tuple (Ct_1, Rt, Ct_2) where Ct_1, Ct_2 are in $Types(\mathcal{O})$ and Rt is a role type such that:*

- *If $\forall R.C \in Ct_1$ and $R \in Rt$ then $C \in Ct_2$.*
- *If $\forall R.C \in Ct_2$ and $Inv(R) \in Rt$ then $C \in Ct_1$.*
- *If $\leq 0R.C \in Ct_1$ and $R \in Rt$ then $\dot\neg C \in Ct_2$.*
- *If $\leq 0R.C \in Ct_2$ and $Inv(R) \in Rt$ then $\dot\neg C \in Ct_1$.*

Definition 9 (Instantiation function). *Given a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}) \models \mathcal{O}$ the function $link(e, f)$ on the set of pairs of domain elements is defined as $(ctype(e), rtype(e, f), ctype(f))$ where ctype is the concept type function and rtype is the role type function.*

Observe that *link* is a well-defined function since *ctype* and *rtype* are well-defined. It is important to emphasize that links inherit the key property of types that any ordered pair of domain elements is an instance of one and only one link. We use $Links(\mathcal{O})$ to denote the set of all links L such that there exists an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{O}$ and $e, f \in \Delta^{\mathcal{I}}$ with $link(e, f) = L$.

Now we can define the notion of consistency between a link and an entity.

Definition 10 (Consistency with an Entity). *A link* (Ct_1, Rt, Ct_2) *is consistent with an entity* ϕ *w.r.t. the ontology* $\mathcal{O}$ *(denoted as* $L \dashv\vdash \phi$*) if:*

- *If* ϕ *is a role* R *and* $R \in Rt$*,*
- *If* ϕ *is a concept* C *and* $C \in Ct_1$*,*
- *If* ϕ *is a concept assertion* $C(e)$ *and* $\{e, C\} \in Ct_1$*,*
- *If* ϕ *is a role assertion* $R(e, f)$ *and* $(e \in Ct_1$ *and* $R \in Rt$ *and* $f \in Ct_2)$ *or* $(f \in Ct_1$ *and* $R \in Rt$ *and* $e \in Ct_2)$*.*

We now proceed to probabilistic models in P-$\mathcal{SHIQ}^+$.

Definition 11 (Probabilistic Interpretation, Probability of an Entity). *A probabilistic interpretation* Pr *of a probabilistic ontology* $PO = (\mathcal{O}, \mathcal{P})$ *is a probability function on* $Links(\mathcal{O})$ *(a mapping* $Pr : Links(\mathcal{O}) \rightarrow [0, 1]$ *s.t.* $\sum_{L \in Links(\mathcal{O})} Pr(L) = 1$*.*

The probability of an entity ϕ *denoted as* $Pr(\phi)$ *is the sum* $\sum_{L \in Link(\mathcal{T}), L \dashv\vdash \phi} Pr(L)$*. Probability of the conjunction of two entities* $Pr(\psi \wedge \phi)$ *is the sum* $\sum_{L \in Links(\mathcal{O}), L \dashv\vdash \phi, L \dashv\vdash \psi} Pr(L)$*. Conditional probability* $Pr(\psi|\phi)$ *is an abbreviation of* $\frac{Pr(\psi \wedge \phi)}{Pr(\phi)}$*.*

As long as any fixed *ordered* pair of domains elements is an instance of one and only one link, we can compute the probability of an entity as a sum of probabilities of all links which are consistent with it (intuitively, links are disjoint, that is, do not share elements, and exhaustive i.e., cover the space of all pairs). For example, the probability of a role is a total probability of all links in which the ordered pair of individuals is related via this role (informally this means "the probability that a randomly selected pair is related via the role").

Definition 12 (Probabilistic Satisfiability, Logical Entailment). *A probabilistic interpretation* Pr *satisfies (or is a model of) a P-*$\mathcal{SHIQ}^+$ *formula* $(\psi|\phi)[l, u]$ *(denoted as* $Pr \models \phi$*) if* $Pr(\psi|\phi) \in [l, u]$*.* Pr *satisfies a collection of P-*$\mathcal{SHIQ}^+$ *formulas if it satisfies all of them.* Pr *satisfies an ontology* $(\mathcal{O}, \mathcal{P})$ *if* $Pr \models \mathcal{P}$*.*

A P-$\mathcal{SHIQ}^+$ *formula* $(\psi|\phi)[l, u]$ *is a logical consequence of a P-*$\mathcal{SHIQ}^+$ *ontology* PO *if it is satisfied by all probabilistic models of* PO*. It is a tight logical consequence of* PO *if* l *(resp.* u*) is the minimum (resp. the maximum) over all* $Pr \models PO$ *such that* $Pr(\phi) > 0$*.*

Problems of deciding probabilistic satisfiability (PSAT) and computing tight logical entailment (TLogEnt) are stated equivalently to P-$\mathcal{SHIQ}$ (see [4]).

One can verify that P-$\mathcal{SHIQ}^+$ is a fragment of $FOPL_2$ by adapting the translation of P-$\mathcal{SHIQ}$ to $FOPL_2$. All that has to be changed in the proof of Theorem 1 is the construction of states so that they correspond to links. Finally we present a few examples of the P-$\mathcal{SHIQ}^+$ advantages over P-$\mathcal{SHIQ}$:

Example 4 (Combining classical and probabilistic individuals). Consider the ontology from the Example 2. Now both occurrences of *ball* denote the same individual. The probability of $Heavy(ball)$ is defined as: $Pr(Heavy(ball)) = \sum_{I \models Heavy(ball)} Pr(I)$. Consider the set of links that are consistent with $ball$. All of them have Big in the first concept type and at least 90% also have $Dense$. This means that at least 90% of them have $Heavy$ in the first concept type (by the definition of types), therefore the result is [0.9.1] as it should be.

Example 5 (Probabilistic role assertions). Ontology from the Example 3 is a well-formed P-$\mathcal{SHIQ}^+$ ontology. The answer to the query $(john : (Tall \sqcap \exists friendOf.\neg Tall)[?, ?]$ is $[0.5, 0.9]$.

Example 6 (Probabilistic role hierarchies). It is possible to assert in P-$\mathcal{SHIQ}^+$ that at least 60% of classmates are friends: $(friendOf|classmateOf)[0.60, 1]$.

3.3 Possible Further Extensions

P-$\mathcal{SHIQ}^+$ does address some important limitations of P-$\mathcal{SHIQ}$ but it is still not a fully appropriate formalism for representing and reasoning with statistical knowledge. Its semantics is based on possible worlds (links) which results in counterintuitive inference in some situations (see Example 1). No further extension of the possible world semantics can solve this issue, so it is natural to investigate the possibilities of using domain probability distributions.

At the moment it is not entirely clear whether the improper handling of statistics turns out to be critical in practical modeling or not. To be visible the problem requires some specific interaction between interpretations of concepts and probabilistic axioms. However, designers of probabilistic ontologies must be aware of it. We are planning to investigate this issue by extending the previously created BRCA ontology [5]. If the issue happens to be critical then there might several ways to rectify it. The two main alternatives are Bacchus' L_p logic with domain based semantics followed by applying belief functions to adjust subjective beliefs basing on the general statistics [10, 11], and Halpern's Type 3 semantics which combines features of domain based and prossible world based models [3].

4 Conclusion

In this paper we presented a new look at the probabilistic description logic P-$\mathcal{SHIQ}$ as a fragment of probabilistic first-order logic. We gave a formal syntactic translation of P-$\mathcal{SHIQ}$ knowledges bases into FOPL_2 theories and proved its faithfulness. This brought an extra insight into P-$\mathcal{SHIQ}$, most importantly, into its limitations, both those that can be addressed by extending the existing possible world based semantics and those which cannot.

We then proceeded to address the first type of limitations and presented an extended logic P-$\mathcal{SHIQ}^+$ which adds a few important capabilities to the existing arsenal. Namely, it enables full probabilistic role assertions, probabilistic role hierarchies and eliminates the unnatural separation between classical and probabilistic individuals. The logic is still decidable and even has the same worst case complexity as P-$\mathcal{SHIQ}$. Our next goal is to implement P-$\mathcal{SHIQ}^+$ by applying techniques that proved useful for P-$\mathcal{SHIQ}$ [12].

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